

# Vorticity statistics in the direct cascade of two-dimensional turbulence

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For the steady-state direct cascade of two-dimensional ( $2d$ ) Navier-Stokes turbulence, we derive analytically the probability of strong vorticity fluctuations. When  $\varpi$  is the vorticity coarse-grained over a scale  $R$ , the probability density function (PDF),  $\mathcal{P}(\varpi)$ , has an asymptotic behavior  $\ln \mathcal{P} \sim -\varpi/\varpi_{rms}$  at  $\varpi \gg \varpi_{rms} = [H \ln(L/R)]^{1/3}$ , where  $H$  is the enstrophy flux and  $L$  is the pumping length. Therefore, the pdf has exponential tails and it is self-similar, i.e. it can be presented as a function of a single argument,  $\varpi/\varpi_{rms}$ , in distinction from other known direct cascades.

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## I. INTRODUCTION

Turbulence is a paradigmatic far-from-equilibrium state of matter and the central question of physics of turbulence is that of universality: how much one needs to know about an external forcing (or initial data for decaying turbulence) to predict the basic features of flow statistics. A related question is that of symmetries of the statistics, particularly whether scale invariance appears for the scales distant from  $L$ , where turbulence is excited [1, 2]. One distinguishes direct and inverse cascades occurring at the scales much smaller or much larger than  $L$ , respectively. Data suggest that the statistics of inverse cascades are scale invariant [2–4]. For example, the probability density function (PDF)  $\mathcal{P}$  of  $\varpi$  that is the vorticity  $\omega$  coarse-grained over a scale  $R$  is empirically found to be a function of a single variable rather than two in  $2d$  inverse energy cascade:  $\mathcal{P}(\varpi) = \varpi^{-1} f(\varpi R^{-2/3})$  [7–10]. Such self-similarity was never observed in direct cascades for whatever small  $R/L$ , the probability distributions of  $\varpi$  change their forms with varying the ratio  $R/L$  [1–4].

One way to explain this profound difference is to argue that fluid motions are slower when the scales are larger. As an inverse cascade proceeds, it has an ample time to be effectively averaged over the small-scale fluctuations including those of the pumping, whose only memory left is the value of flux it generates. On the contrary, small-scale fast fluctuations in a direct cascade stay sensitive to the statistics of fluctuations at larger scales [11], nonlinearity then enhances the effect of fluctuations down the cascade so that the small-scale statistics is dominated by rare strong fluctuations.

One can also explain the difference between the direct and inverse cascades using the Lagrangian language. Correlation functions are accumulated along Lagrangian trajectories (for the forced turbulence) or originate from initial data transported along the Lagrangian trajectories (for the decaying turbulence). Correlation functions are then proportional to the time the trajectories spend within the volume of size  $L$ . When the trajectories approach each other back in time, correlations appear at larger and larger scales, which corresponds to an inverse cascade. In this case, two-particle behavior effectively determines evolution of multi-particle configurations and the second moment exponent determines the scaling of higher moments. Just the opposite, direct cascades correspond to trajectories separating back in time, one then relates the breakdown of scale-invariance at vanishing viscosity to non-uniqueness of explosively separating trajectories in a non-smooth velocity field; exponents of higher moments are then related to the law of decay of the fluctuations of the shapes of multi-particle configurations which depend on the number of particles [3, 12]. Say, for the passive scalar, the exponents are independent of the scalar pumping but are dependent on the mixing velocity statistics [13]. Still, one needs to know an infinite number of forcing-related parameters to predict the scalar statistics at however small scales. For the direct cascades in Navier-Stokes turbulence and

similar nonlinear problems, it is not even known whether the exponents of the velocity moments are universal or not.

Prior knowledge was based on experimental and numerical data, the only analytical results were obtained for passive fields in synthetic flows [3, 14–17]. Here, for the first time, the vorticity PDF tail is analytically derived from the equation of motion.

We consider the direct (enstrophy) cascade of  $2d$  turbulence [18–20], which in Lagrangian terms is peculiar since it corresponds to an exponential separation of trajectories. Indeed, the physical mechanism of the cascade is that pumping-produced vorticity blobs are deformed by the flow into thin streaks: stretched in one direction and contracted in another one until viscosity dissipates them, like for a passive scalar. An important distinction from the passive scalar is that the vorticity  $\omega$  is related to the flow velocity  $\mathbf{v}$ :  $\omega = \text{curl } \mathbf{v}$ . Constancy of the enstrophy flux over scales,  $\langle (\mathbf{v}_1 \cdot \nabla_1 + \mathbf{v}_2 \cdot \nabla_2) \omega_1 \omega_2 \rangle = \text{const}$ , suggests the scaling  $|\mathbf{v}(\mathbf{r}) - \mathbf{v}(0)| \propto r$  i.e. spatially smooth velocity. In a steady state, the enstrophy dissipation  $\nu |\nabla \omega|^2$  must stay finite in the inviscid limit  $\nu \rightarrow 0$ . The velocity then cannot be perfectly smooth, but the possible singularities are no stronger than logarithmic [21, 22]. If one assumes self-similarity in a sense that the PDF of the coarse-grained vorticity is  $\mathcal{P}(\varpi) = \varpi^{-1} f[\varpi^a / \ln(L/R)]$ , then the flux constancy requires  $a = 3$  [18, 21, 23].

There are some consequences of the self-similarity. Say, the enstrophy transfer time through a given scale  $R$ , determined by the stretching/contraction rate, can be estimated as a turn-over time or an inverse vorticity at this scale. On the one hand, that time decreases with the scale as  $\ln^{-1/3}(L/R)$ , which would suggest that the small-scale statistics is sensitive to the statistics at larger scales. On the other hand, the total time of enstrophy transfer from  $L$  down to the viscous scale  $\eta$  diverges  $\propto \ln^{2/3}(L/\eta)$  as  $\eta \rightarrow 0$ . Particle trajectories are then expected to separate exponentially rather than explosively and stay unique even in the inviscid limit, that makes self-similarity plausible, according to the above Lagrangian arguments. Note that von Neumann [24] and Kraichnan [18] argued that an infinite number of vorticity conservation laws can make the vorticity cascade non-universal, yet Falkovich and Lebedev later argued that the fluxes of higher vorticity invariants must be irrelevant due to the phenomenon of “distributed pumping” [21]. Recently, self-similarity breakdown was found empirically for the vorticity isolines, which are conformal invariant in the inverse cascade, while in the direct cascade they are not scale-invariant but multi-fractal with the fractal dimension  $3/2$  and higher dimensions saturating at 1 [5, 6] (that may be related to strain persistence that leads to vorticity organized in long thin streaks). That makes it natural to expect that the bulk vorticity statistics is not self-similar as well.

The present work is devoted to analytical description of a single-time vorticity statistics in the steady-state  $2d$  turbulence in the direct cascade. We analytically derive the non-Gaussian tail of the PDF of  $\varpi$ , that is the vorticity coarse-grained over the scale  $R$ , in the direct (enstrophy) cascade. We show that the tail is exponential,

$$\ln \mathcal{P}(\varpi) \sim - \frac{|\varpi|}{[H \ln(L/R)]^{1/3}}, \quad (1.1)$$

for a driving force with a finite correlation time. In particular, Eq. (1.1) shows that the vorticity PDF is self-similar, i.e. it can be presented as  $\mathcal{P}(\varpi) = \varpi^{-1} f[\varpi^3 / \ln(L/R)]$ . Moreover, up to the order-unity factor, the tail (1.1) is determined by a single parameter,  $H$ , that is the flux of the squared vorticity, which also determines the  $\varpi_{rms}$  i.e. the bulk of the pdf. To obtain the single-point vorticity PDF, the ratio  $L/R$  should be substituted by  $\sqrt{Re}$  in Eq. (1.1).

The structure of our paper is as follows. ....

## II. BASIC EQUATIONS

The incompressible  $2d$  Euler equation can be written for the vorticity  $\omega = \partial_x v_y - \partial_y v_x$ :

$$\frac{\partial \omega}{\partial t} + (\mathbf{v} \nabla) \omega = \phi. \quad (2.1)$$

Here  $\phi$  is curl of the external force  $\mathbf{f}$  exciting the turbulence:  $\phi = \partial_x f_y - \partial_y f_x$ . The viscous term is omitted in (2.1), which means that we consider flow variations on scales much larger than the viscous scale  $\eta$ . We shall describe the flow in the Lagrangian reference frame attached to a fluid particle placed at the origin, such that  $\mathbf{v}(\mathbf{0}) = \mathbf{0}$ . Then the velocity is expressed via the vorticity as

$$v_\alpha(\mathbf{r}) = -\epsilon_{\alpha\beta} \int \frac{d^2 r'}{2\pi} \left( \frac{r_\beta - r'_\beta}{|\mathbf{r} - \mathbf{r}'|^2} + \frac{r'_\beta}{|\mathbf{r}'|^2} \right) \omega(\mathbf{r}'). \quad (2.2)$$

The pumping  $\phi$  is assumed to be a random Gaussian field spatially correlated on the scale  $L$  and short correlated in time. Then its variance is  $\langle \phi(0, \mathbf{0}) \phi(t, \mathbf{r}) \rangle = \delta(t) \chi(r)$ , where  $\chi(r)$  rapidly tends to zero as  $r$  exceeds  $L$ . As we shall see below, the processes that contribute to the vorticity PDF tails take a long time which allows effective averaging over forcing so that our results are asymptotically valid for any forcing with a finite correlation time.

The statistics of the flow can be examined within the framework of Martin-Siggia-Rose formalism [25–28] so that all the averages (correlation functions) characterizing the flow are calculated as functional integrals,  $\int \mathcal{D}p \mathcal{D}\omega \exp(i\mathcal{I}) \dots$ , with the effective action

$$\mathcal{I} = \int dt d^2 r p(\mathbf{r}) \left[ \partial_t \omega + \mathbf{v} \nabla \omega + \frac{i}{2} \int d^2 r' \chi(|\mathbf{r} - \mathbf{r}'|) p(\mathbf{r}') \right]. \quad (2.3)$$

Here  $p$  is an auxiliary field introduced to put the equation of motion (2.1) into the exponent. Since the action (2.3) contains a cubic term originating from the nonlinear term in Eq. (2.1), one is unable to calculate the functional integrals explicitly. Nor one is able to treat the third-order term by a perturbation theory, since there is no small parameter in the expansion. In other words, we deal with the theory where the coupling is strong. What allows for an analytic description is that we consider rare strong fluctuations i.e. describe tails of the vorticity PDF.

We consider the PDF of  $\varpi$ , that is the vorticity  $\omega$  coarse-grained over a scale  $R$  from the interval of the direct cascade, that is we assume that  $R$  is much smaller than the pumping scale  $L$  but larger than the viscous length  $\eta$ . A general strategy to find tails of the PDF,  $\mathcal{P}(\varpi)$ , is to calculate the corresponding functional integral in the saddle-point approximation utilizing the ratio  $\varpi/\varpi_{rms}$  as a large parameter. The way to do that is the so-called instanton formalism adapted for turbulence problems [17, 30–33]. In this way, one looks for an extremum of the action (2.3), defined by the instanton (extremum) equations  $\delta\mathcal{I}/\delta\omega = 0 = \delta\mathcal{I}/\delta p$  with appropriate boundary conditions. Both the action and the measured quantity  $\omega$  are invariant with respect to rotations and so are instanton equations and their boundary conditions. However, axial symmetry turns nonlinear terms in the instanton equations into zero killing dynamics. In other words, a “naive instanton” is meaningless. The physical reason is quite transparent: there is neither stretching nor contraction for axially symmetric flows so that the force can pump the vorticity forever. That means that flow realizations that determine a given large value of  $\varpi$  must have their axial symmetry broken. We establish below that the angle-dependent part of the vorticity realizations remains much smaller than the isotropic part during most of the evolution (by virtue of the large parameter  $\varpi/\varpi_{rms}$ ). That will allow us to integrate over angular degrees of freedom (in the Gaussian approximation) and obtain a renormalized action for the zero harmonic  $\omega_0$ . Moreover, we show that only the second angular harmonic provides

for the relevant renormalization by virtue of the large parameter  $\ln(L/r)$ . We then find the new (effectively axially symmetric) instanton that corresponds to the renormalized action and gives the tail of the coarse-grained vorticity PDF,  $\mathcal{P}(\varpi)$ .

### III. SEPARATION OF HARMONICS

We use polar coordinates,  $x = r \cos \varphi$ ,  $y = r \sin \varphi$ , and expand the fields  $\omega$  and  $p$  over the angular harmonics:

$$\omega(t, \mathbf{r}) = \sum \omega_m(t, r) \exp(im\varphi), \quad 2\pi p(t, \mathbf{r}) = \sum p_m(t, r) \exp(im\varphi). \quad (3.1)$$

Then the effective action (2.3) splits into a number of terms  $\mathcal{I} = \mathcal{I}_0 + \sum_{m>0} (\mathcal{I}_m + \mathcal{I}_{-m} + \mathcal{I}_{mm} + \mathcal{I}_{im}) + \mathcal{I}_3$ , where  $\mathcal{I}_0$  contains only the zero harmonics  $\omega_0, p_0$ . The last term  $\mathcal{I}_3$  is a sum of the third order terms containing harmonics with  $m \neq 0$ , it is neglected in what follows, which is justified below. The terms quadratic in  $p_m, \omega_m$  are written as follows

$$\mathcal{I}_0 = \int dt dr r p_0 \partial_t \omega_0 + \frac{i}{2} \int dt dr r dr' r' \chi_0(r, r') p_0(r) p_0(r'), \quad (3.2)$$

$$\mathcal{I}_m = \int dt dr r p_{-m} \{ \partial_t \omega_m + v_0 i m \omega_m / r + \partial_r \omega_0 v_{rm} \}, \quad (3.3)$$

$$\mathcal{I}_{im} = - \int dt dr r \partial_r p_0 (v_{r,m} \omega_{-m} + v_{r,-m} \omega_m), \quad (3.4)$$

$$\mathcal{I}_{mm} = i \int dt dr r dr' r' \chi_m(r, r') p_{-m}(r) p_m(r'), \quad (3.5)$$

$$\chi_m(r, r') = \int \frac{d\varphi}{2\pi} e^{im\varphi} \chi \left( \sqrt{r^2 + r'^2 - 2rr' \cos \varphi} \right). \quad (3.6)$$

Here  $v_0$  in Eq. (3.3) is related to  $\omega_0$  via the equation  $\omega_0 = v_0 / r + \partial v_0 / \partial r$ . Our goal now is to derive an effective action for the zero harmonic,  $\mathcal{I}_0 + \Delta \mathcal{I}$  by integrating over all the other harmonics,

$$\exp(i\mathcal{I}_0 + i\Delta \mathcal{I}) = \int \prod_{m>0} \mathcal{D}\omega_{\pm m} \mathcal{D}p_{\pm m} \exp(i\mathcal{I}). \quad (3.7)$$

The integration is Gaussian if to neglect third order terms in  $p_m$  and  $\omega_m$ , as explained above. Then  $\Delta \mathcal{I} = \sum \Delta \mathcal{I}_m$  where

$$e^{i\Delta \mathcal{I}_m} = \int \mathcal{D}\omega_{\pm m} \mathcal{D}p_{\pm m} e^{i\mathcal{I}_m + i\mathcal{I}_{-m} + i\mathcal{I}_{mm} + i\mathcal{I}_{im}}. \quad (3.8)$$

If  $\mathcal{I}_{im} = 0$  then the expression (3.8) is the normalization integral that is equal to unity due to causality [28]. Therefore one can write

$$\Delta \mathcal{I} = \sum_{m>0} \sum_{n=1}^{\infty} \frac{i^{n-1}}{n!} \langle (\mathcal{I}_{im})^n \rangle_c, \quad (3.9)$$

where the angular brackets mean integration over  $\omega_{\pm m}$  and  $p_{\pm m}$  with the weight  $\exp(i\mathcal{I}_m + i\mathcal{I}_{-m} + i\mathcal{I}_{mm})$  and the subscript  $c$  means an irreducible average (represented by connected diagrams).

Consistently considering small fluctuations (as in neglecting  $\mathcal{I}_3$ ) we take only the term with  $n = 1$  in Eq. (3.9). We shall justify it later by observing that the angular part remains small during the

build-up of the strong fluctuation that we consider. Therefore, the main object, that contributes to the  $n = 1$  term and need to be examined, is the pair correlation function

$$\langle \omega_m(t, r) \omega_{-m}(t', r') \rangle = F_m(t, t'; r, r'). \quad (3.10)$$

The simultaneous pair correlation function satisfies the equation

$$\begin{aligned} \partial_t F_m(t, t, r_1, r_2) + im \left[ \frac{v_0(r_1)}{r_1} - \frac{v_0(r_2)}{r_2} \right] F_m(r_1, r_2) \\ - \partial_r \omega_0(r_1) \frac{i}{2} \int dr r^2 \frac{u_1^{|m+1|} - u_1^{|m-1|}}{|r^2 - r_1^2|} F_m(r, r_2) \\ + \partial_r \omega_0(r_2) \frac{i}{2} \int dr r^2 \frac{u_2^{|m+1|} - u_2^{|m-1|}}{|r^2 - r_2^2|} F_m(r_1, r) = \chi_m(r_1, r_2), \end{aligned} \quad (3.11)$$

where  $u_{1,2} = \min\{r/r_{1,2}, r_{1,2}/r\}$ . One should treat separately the first angular harmonic, with  $m = \pm 1$ .

#### A. Logarithmic approximation

Let us pass to the logarithmic variable  $\xi = \ln(r/L)$ , where  $L$  is the pumping length. We are interested in small scales,  $r \ll L$  where  $|\xi| \gg 1$ . We consider only the leading contributions in terms of large  $|\xi|$ . In this case, only the terms with  $m = 2$  are relevant in Eq. (3.9) since the integration in the expressions (2.2, 3.11) is logarithmic only for them, as has been noticed already in [21], the feature is likely related to peculiarity of elliptic vortices in straining flows [29]. Other harmonics behave as  $r^{m-2}$  for  $m > 2$  i.e. are suppressed exponentially in terms of the large logarithm  $\xi$ . In the logarithmic variables,  $\chi_m(\xi_1, \xi_2)$  for  $m \neq 0$  are nonzero only if both  $|\xi_1|, |\xi_2| \lesssim 1$ , since the integral (3.6) is zero for  $r_1 + r_2 < L$  and decays as  $L/\sqrt{r_1 r_2} = \exp[-(\xi_1 + \xi_2)/2]$  for  $r_1, r_2 \rightarrow \infty$ . That means that one can approximate  $\chi_m(\xi_1, \xi_2) \approx H_m \delta(\xi_1) \delta(\xi_2)$  for  $m \neq 0$ . The zeroth harmonics can be taken as  $\chi_0(\xi_1, \xi_2) = H \theta(-\xi_1) \theta(-\xi_2)$  where  $\theta$  is the step function.

Next, we pass to the field  $q(\xi) = r^2 p_0(r)$ . Then the bare action (3.2) for the zeroth harmonics is rewritten as

$$\mathcal{I}_0 = \int dt d\xi q \partial_t \omega_0 + \frac{i}{2} \int dt d\xi_1 d\xi_2 \chi_0(\xi_1, \xi_2) q(\xi_1) q(\xi_2). \quad (3.12)$$

The correction  $\Delta \mathcal{I}$  (3.9) in the main approximation (taking into account only the term with  $n = 1$ ,  $m = 2$ ) can be written as

$$\Delta \mathcal{I} \approx i \int dt d\xi q(t, \xi) \int_\xi d\zeta [F_2(t, t; \zeta, \xi) - F_2(t, t; \xi, \zeta)]. \quad (3.13)$$

Here the function  $F_2$  has to be treated as a functional of  $\omega_0$  to be extracted from the equation (3.11) for  $m = 2$ .

In terms of the logarithmic variable  $\xi$  the equation (3.11) is rewritten as

$$\begin{aligned} \partial_t F_2(\xi_1, \xi_2) + 2i[w(\xi_1) - w(\xi_2)] F_2(\xi_1, \xi_2) - \chi_2(\xi_1, \xi_2) = \\ \frac{i \partial_\xi \omega_0(\xi_2)}{2} \left[ \int_{\xi_2}^\infty d\xi F_2(\xi_1, \xi) + \int_{-\infty}^{\xi_2} d\xi e^{4(\xi - \xi_2)} F_2(\xi_1, \xi) \right] \\ - \frac{i \partial_\xi \omega_0(\xi_1)}{2} \left[ \int_{\xi_1}^\infty d\xi F_2(\xi, \xi_2) + \int_{-\infty}^{\xi_1} d\xi e^{4(\xi - \xi_1)} F_2(\xi, \xi_2) \right], \end{aligned} \quad (3.14)$$

where we denoted  $w(\xi) \equiv v_0/r = \int_{-\infty}^{\xi} d\xi' \exp[2(\xi' - \xi)]\omega_0(\xi')$ . In the main logarithmic approximation we get from Eq. (3.14)

$$\begin{aligned} & \partial_t F_2(\xi_1, \xi_2) + i\omega_0(t, \xi_1)F_2(\xi_1, \xi_2) + \frac{i}{2}\partial_{\xi}\omega_0(t, \xi_1) \int_{\xi_1}^{\infty} d\xi F_2(\xi, \xi_2) \\ & - i\omega_0(t, \xi_2)F_2(\xi_1, \xi_2) - \frac{i}{2}\partial_{\xi}\omega_0(t, \xi_2) \int_{\xi_2}^{\infty} d\xi F_2(\xi_1, \xi) = \chi_2(\xi_1, \xi_2). \end{aligned} \quad (3.15)$$

Putting in Eq. (3.15)  $\xi_1 = \xi_2 = \xi$  and substituting the result into the expression (3.13) one obtains

$$\Delta\mathcal{I} \approx -2 \int dt d\xi \frac{q}{\partial_{\xi}\omega_0} \partial_t F_2(\xi, \xi), \quad (3.16)$$

the term with  $\chi_2$  is neglected since it is small in the interval of the direct cascade.

### B. Eigen functions

The equation (3.15) can be rewritten in the form

$$\partial_t F_2(\xi_1, \xi_2) + i\hat{O}_1 F_2 - i\hat{O}_2 F_2 = \chi_2(\xi_1, \xi_2), \quad (3.17)$$

where

$$\hat{O}f(\xi) = \int d\zeta \left[ \omega_0\delta(\xi - \zeta) + \frac{1}{2}\partial_{\xi}\omega_0\theta(\zeta - \xi) \right] f(\zeta). \quad (3.18)$$

Let us introduce eigen functions of the operator  $\hat{O}$

$$\hat{O}\varphi_{\lambda} = \omega_0(\xi)\varphi_{\lambda}(\xi) + \frac{1}{2}\frac{\partial\omega_0(\xi)}{\partial\xi} \int_{\xi}^{\infty} d\zeta \varphi_{\lambda}(\zeta) = \lambda\varphi_{\lambda}(\xi). \quad (3.19)$$

We assume (in accordance with the answer obtained) that  $\omega_0(\xi)$  monotonically diminishes from some value  $s$  at  $\xi = -\infty$  to zero at  $\xi = +\infty$ . Then a set of the eigen functions of the operator  $\hat{O}$  can be written as

$$\varphi_{\lambda} = \theta(\omega_0 - \lambda)2\partial_{\xi}\sqrt{\omega_0 - \lambda} = \frac{\theta(\omega_0 - \lambda)}{\sqrt{\omega_0 - \lambda}}\partial_{\xi}\omega_0, \quad (3.20)$$

where  $\theta$  is the step function and  $0 < \lambda < s$ . The functions (3.20) are the right eigenfunctions of the operator  $\hat{O}$ . Analogously, one can define the left eigenfunctions:

$$\phi_{\mu}(\xi) = \frac{1}{2\pi} \lim_{\epsilon \rightarrow 0} \text{Re} [\mu - \omega_0(\xi) + i\epsilon]^{-3/2}, \quad (3.21)$$

where, again,  $0 < \mu < s$ . The functions (3.21) satisfy the equation

$$\omega_0(\xi)\phi_{\mu}(\xi) + \frac{1}{2} \int_{-\infty}^{\xi} d\zeta \partial_{\zeta}\omega_0\phi_{\mu}(\zeta) = \mu\phi_{\mu}(\xi). \quad (3.22)$$

The only thing which is important for what follows is the orthogonality and normality of the right and left eigenfunctions. The factors in (3.20,3.21) are chosen to ensure the normalization condition

that can be checked directly:

$$\begin{aligned} \int d\xi \varphi_\lambda(\xi) \phi_\mu(\xi) &= -\frac{1}{2\pi} \lim_{\epsilon \rightarrow 0} \operatorname{Re} \int_\lambda^s \frac{d\omega_0}{\sqrt{\omega_0 - \lambda}} \frac{1}{(\mu - \omega_0 + i\epsilon)^{3/2}} \\ &= \frac{1}{\pi} \sqrt{s - \lambda} \lim_{\epsilon \rightarrow 0} \operatorname{Re} (\lambda - \mu - i\epsilon)^{-1} (\mu + i\epsilon - s)^{-1/2} = \delta(\mu - \lambda), \end{aligned} \quad (3.23)$$

provided  $0 < \mu, \lambda < s$ . One can check completeness of the set (3.20,3.21):

$$\int_0^s d\lambda \phi_\lambda(\xi) \varphi_\lambda(\zeta) = \delta(\xi - \zeta). \quad (3.24)$$

The check is reduced to an integral analogous to one (3.23).

Let us expand  $F_2$  over the eigenfunctions (3.20),

$$F_2(t, \zeta_1, \zeta_2) = \int d\lambda_1 d\lambda_2 \Phi(t, \lambda_1, \lambda_2) \varphi_{\lambda_1}(\zeta_1) \varphi_{\lambda_2}(\zeta_2), \quad (3.25)$$

where

$$\Phi(\lambda_1, \lambda_2) = \int d\xi_1 d\xi_2 \phi_{\lambda_1}(\xi_1) \phi_{\lambda_2}(\xi_2) F_2(\xi_1, \xi_2). \quad (3.26)$$

Now we take into account that  $\omega_0 \rightarrow 0$  as  $\xi \rightarrow +\infty$  and obtain:

$$\begin{aligned} &[\partial_t + i(\mu_1 - \mu_2)] \Phi(t, \mu_1, \mu_2) + \int d\lambda_1 \Phi(\lambda_1, \mu_2) J(\mu_1, \lambda_1) \\ &+ \int d\lambda_2 \Phi(\mu_1, \lambda_2) J(\mu_2, \lambda_2) = \int d\xi_1 d\xi_2 \phi_{\mu_1}(\xi_1) \phi_{\mu_2}(\xi_2) \chi_2(\xi_1, \xi_2), \end{aligned} \quad (3.27)$$

where

$$J(\mu, \lambda) = \int d\zeta \phi_\mu(\zeta) \partial_t \varphi_\lambda(\zeta) = - \int d\zeta \partial_t \phi_\mu(\zeta) \varphi_\lambda(\zeta). \quad (3.28)$$

The equation (3.27) is equivalent to Eq. (3.17).

Substituting into the definition (3.28) the explicit expressions (3.20,3.21) we get

$$J(\mu, \lambda) = \frac{1}{\pi} \frac{\partial^2}{\partial \mu^2} \left\{ \theta(\mu - \lambda) \int_\lambda^\mu \frac{d\omega_0 \psi(t, \omega_0)}{\sqrt{(\omega_0 - \lambda)(\mu - \omega_0)}} \right\}. \quad (3.29)$$

Here we introduced the designation

$$\partial_t \omega_0(t, \xi) = \psi(t, \omega_0). \quad (3.30)$$

Note that it follows from the definition (3.30) that

$$\psi' = \frac{\partial \psi(t, \omega_0)}{\partial \omega_0} = \frac{\partial \psi}{\partial \xi} \frac{\partial \xi}{\partial \omega_0} = \partial_t \ln(\partial_\xi \omega_0). \quad (3.31)$$

Performing the substitution  $\omega_0 = \lambda + (\mu - \lambda)x$  we get from Eq. (3.29)

$$\begin{aligned} J(\mu, \lambda) &= \frac{\partial}{\partial \mu} [\delta(\mu - \lambda) \psi(\lambda)] + \frac{1}{2} \delta(\mu - \lambda) \psi'(\lambda) \\ &+ \frac{1}{\pi} \theta(\mu - \lambda) \int_0^1 \frac{dx x^{3/2}}{\sqrt{1-x}} \psi''[\lambda + (\mu - \lambda)x]. \end{aligned} \quad (3.32)$$

We see that  $J$  is the sum of singular terms and the term, which can be written as a regular expansion over  $\mu - \lambda$ .



### C. Initial Condition

As we will see, the instantonic solution for  $\omega_0$  diminishes back in time. Therefore the saddle-point approximation ceases to be correct at a time  $t = t_*$  where  $\omega_0$  is of the order of typical (rms) fluctuation and one should pose the initial condition for the instanton at  $t = t_*$ . We assume that the initial fluctuation (at  $t = t_*$ )  $\omega_0$  is some slow (logarithmic) function of the distances in the region  $|\xi| \lesssim \ln(L/r)$  so that  $\partial_\xi \ln \omega_0 \sim \xi^{-1}$ . Next, we assume that at  $t = t_*$

$$\langle \omega(\mathbf{r}_1) \omega(\mathbf{r}_2) \rangle \sim (H\gamma)^{2/3}, \quad (3.33)$$

$$\begin{aligned} \gamma &= \ln \frac{|\mathbf{r}_1 - \mathbf{r}_2|}{L} = \frac{1}{2} \ln [(r_1^2 + r_2^2 - 2r_1 r_2 \cos \varphi)/L^2] \\ &= \frac{1}{2} \ln \left[ \frac{r_1^2 + r_2^2}{L^2} \right] + \frac{1}{2} \ln \left[ 1 - \frac{2r_1 r_2 \cos \varphi}{r_1^2 + r_2^2} \right]. \end{aligned} \quad (3.34)$$

It is important to stress that, strictly speaking, we cannot derive the second moment (3.33) from the equation of motion. That choice is consistent with the flux relation and, as we show below, is self-consistent with the higher moments described by the PDF tail to be derived.

We now derive the second angular harmonic of the pair correlation function:

$$F_2(t_*, \xi_1, \xi_2) = \int_0^{2\pi} \frac{d\varphi}{2\pi} \exp(-2i\varphi) \langle \omega(\mathbf{r}_1) \omega(\mathbf{r}_2) \rangle. \quad (3.35)$$

If  $|\xi_1|, |\xi_2| \gg 1$  then a power-like function  $f(\psi)$  can be expanded in the ratio of the two last terms in Eq. (3.34). Then one obtains

$$\int_0^{2\pi} \frac{d\varphi}{2\pi} \exp(-2i\varphi) f(\gamma) = -\frac{1}{4} \frac{df}{d\gamma_0} \times \begin{cases} r_2^2/r_1^2 & \text{if } r_1 > r_2, \\ r_1^2/r_2^2 & \text{if } r_1 < r_2, \end{cases} \quad (3.36)$$

where  $\gamma_0 = (1/2) \ln[(r_1^2 + r_2^2)/L^2]$  and we used the relation

$$\int_0^{2\pi} dx \cos(2x) \ln(1 - a \cos x) = \frac{\pi}{a^2} (a^2 - 2 + 2\sqrt{1 - a^2}).$$

We conclude that the function  $F_2(t_*)$  exponentially in  $\xi_1 - \xi_2$  tends to zero as  $|\xi_1 - \xi_2| \equiv |\ln(r_1/L) - \ln(r_2/L)|$  tends to infinity. Therefore it can be estimated as  $F_2(t_*) \sim H^{2/3} |\xi_1|^{-1/3} \delta(\xi_1 - \xi_2)$ .

Now we can analyze the initial value of the function  $\Phi$

$$\Phi(t_*, \lambda_1, \lambda_2) = \int d\xi_1 d\xi_2 \phi_{\lambda_1}(\xi_1) \phi_{\lambda_2}(\xi_2) F_2(t_*, \xi_1, \xi_2). \quad (3.37)$$

Substituting here the above estimate, one obtains

$$\Phi(t_*, \lambda_1, \lambda_2) \sim H^{2/3} \int_{-\infty}^0 d\xi |\xi|^{-1/3} \phi_{\lambda_1}(\xi) \phi_{\lambda_2}(\xi). \quad (3.38)$$

Again, the approximation (3.38) implies  $\lambda_1, \lambda_2 \gg H^{1/3}$ . What is most important is that the integrand of (3.38) has a third-order pole when  $\lambda_1 \rightarrow \lambda_2 \rightarrow \omega_0$ . Integration over  $\xi$  results in the second-order pole:  $\Phi_0(\lambda_1, \lambda_2) \propto (\lambda_1 - \lambda_2)^{-2}$  at  $\lambda_1 \rightarrow \lambda_2$ . Therefore

$$\Phi(t_*, \lambda_1, \lambda_2) \sim \frac{(H\xi)^{2/3}}{\sqrt{\lambda_1 \lambda_2}} \delta'(\lambda_1 - \lambda_2), \quad (3.39)$$

where  $\xi$  is determined by the condition  $\omega_0(t_*, \xi) = \lambda_1 \approx \lambda_2$ .



### D. Adiabatic approximation

The instantonic field  $\omega_0(t, \xi)$  describes an optimal fluctuation that starts from the rms level at the time  $t_*$  and grows to a prescribed large value of  $\varpi$ . After we find below the instanton solution  $\omega_0(t, \xi)$ , we see that its form changes slow on its own rotation timescale  $\omega_0^{-1}$ . In particular, that means that the eigenfunctions (3.20) change slow too, so that one can neglect non-singular term in the expression (3.32). Then

$$J(\mu, \lambda) \rightarrow \frac{\partial}{\partial \mu} [\delta(\mu - \lambda) \psi(\lambda)] + \frac{1}{2} \delta(\mu - \lambda) \psi'(\lambda). \quad (3.40)$$

Substituting the expression (3.40) into Eq. (3.27) and omitting the right-hand side we get

$$[\partial_t + i(\mu_1 - \mu_2)] \Phi(t, \mu_1, \mu_2) + \psi(\mu_1) \frac{\partial}{\partial \mu_1} \Phi + \psi(\mu_2) \frac{\partial}{\partial \mu_2} \Phi + \frac{3}{2} [\psi'(\mu_1) + \psi'(\mu_2)] \Phi = 0. \quad (3.41)$$

The equation (3.41) can be solved by the method of characteristics. The equation for the characteristics reads

$$\frac{d\mu}{dt} = \psi(t, \mu) = \partial_t \omega_0(t, \xi), \quad (3.42)$$

where  $\xi$  is a function of  $\mu$  to be extracted from the condition  $\mu = \omega_0$ . An obvious solution of the equation (3.42) is

$$\mu(t) = \omega_0(t, \xi), \quad (3.43)$$

where  $\xi$  plays the role of the marker of the characteristic. A solution of the equation (3.41) is written as

$$\begin{aligned} \Phi(t, \mu_1, \mu_2) &= \Phi[t_*, \mu_1(t_*), \mu_2(t_*)] \\ &\times \exp \left( \int_{t_*}^t ds \left\{ -i\mu_1(s) + i\mu_2(s) - \frac{3}{2} \psi'[s, \mu_1(s)] - \frac{3}{2} \psi'[s, \mu_2(s)] \right\} \right). \end{aligned} \quad (3.44)$$

Using Eq. (3.31) we get

$$\begin{aligned} \Phi(t, \mu_1, \mu_2) &= \left[ \frac{\partial_{\zeta} \omega_0(t_*, \zeta_1)}{\partial_{\zeta} \omega_0(t, \zeta_1)} \frac{\partial_{\zeta} \omega_0(t_*, \zeta_2)}{\partial_{\zeta} \omega_0(t, \zeta_2)} \right]^{3/2} \Phi[t_*, \mu_1(t_*), \mu_2(t_*)] \\ &\times \exp \left\{ \int_{t_*}^t ds [-i\mu_1(s) + i\mu_2(s)] \right\}, \end{aligned} \quad (3.45)$$

where the variables  $\zeta_1, \zeta_2$  have to be extracted from the relations  $\mu_1 = \omega_0(t, \zeta_1), \mu_2 = \omega_0(t, \zeta_2)$ .

Now we can estimate a role of the regular contribution omitted in Eq. (3.40), see Eq. (3.32). Substituting the expression (3.45) into Eq. (3.27) we then conclude that an integration over  $\lambda_1$  or over  $\lambda_2$  in the omitted terms is determined mainly by the oscillating factor in Eq. (3.45). Then  $\int d\lambda_1 \rightarrow (t - t_*)^{-1}$  (and the same for the integration over  $\lambda_2$ ) and therefore the omitted terms in the equation (3.27) can be estimated as

$$\frac{1}{t - t_*} \psi''(\mu_1) \Phi(\mu_1, \mu_2).$$

Comparing the term with Eq. (3.41), we conclude, that the omitted terms have an additional small factor  $[(t - t_*)\omega_0]^{-1}$  and therefore the approximation leading to Eq. (3.41) is correct.

Let us calculate  $F_2(\xi_1, \xi_2)$  using the expression (3.45):

$$\begin{aligned}
F_2(t, \xi_1, \xi_2) &= \int d\mu_1 d\mu_2 \frac{\theta[\omega(t, \xi_1) - \mu_1]}{\sqrt{\omega(t, \xi_1) - \mu_1}} \frac{\theta[\omega(t, \xi_2) - \mu_2]}{\sqrt{\omega(t, \xi_2) - \mu_2}} \\
&\quad \times \partial_\xi \omega_0(t, \xi_1) \partial_\xi \omega_0(t, \xi_2) \left[ \frac{\partial_\zeta \omega_0(t_*, \zeta_1)}{\partial_\zeta \omega_0(t, \zeta_1)} \frac{\partial_\zeta \omega_0(t_*, \zeta_2)}{\partial_\zeta \omega_0(t, \zeta_2)} \right]^{3/2} \\
&\quad \times \Phi[t_*, \omega_0(t_*, \zeta_1), \omega_0(t_*, \zeta_2)] \exp \left\{ \int_{t_*}^t ds [-i\omega_0(s, \zeta_1) + i\omega_0(s, \zeta_2)] \right\}.
\end{aligned} \tag{3.46}$$

Recall, that  $\zeta_1$  and  $\zeta_2$  have to be extracted from the relations  $\mu_1 = \omega_0(t, \zeta_1)$ ,  $\mu_2 = \omega_0(t, \zeta_2)$ . After substituting (3.39) into (3.46), in the integral over  $\mu_2$  we keep only the pole term at  $\mu_2 \rightarrow \mu_1$ ,  $\zeta_2 \rightarrow \zeta_1$ ; differentiating the exponent gives the  $t - t_*$  factor because of the slowness of the instanton. Passing then from the integration over  $\mu_1$  to ones over  $\zeta$  we obtain

$$\begin{aligned}
F_2(t, \xi, \xi) &\sim H^{2/3} (t - t_*) [\partial_\xi \omega_0(t, \xi)]^2 \int d\zeta \zeta^{2/3} \partial_\zeta \ln \omega_0(t_*, \zeta) \frac{\theta[\omega_0(t, \xi) - \omega_0(t, \zeta)]}{\omega_0(t, \xi) - \omega_0(t, \zeta)} \\
&\sim (t - t_*) (\partial_\xi \omega_0) H^{2/3} \xi^{-1/3} \ln(\omega_0 / \partial_\xi \omega_0).
\end{aligned} \tag{3.47}$$

Here we used  $\partial_\zeta \ln \omega_0(t_*, \zeta) \simeq 1/\zeta$ , it is equivalent to assuming that the initial  $\omega_0(\xi)$  is a power-like function of the logarithm  $\xi$ . The logarithmic divergence in (3.47) is cut off due to a finite (order-unity) width of  $F_2(t_*, \xi_1, \xi_2)$  over  $\xi_1 - \xi_2$ .

Now we turn to the pumping contribution:

$$\Phi(\mu, \lambda, t) = \int_0^{t-t_*} d\tau \exp[-i(\mu - \lambda)(t - t_* - \tau)] \Xi(\mu, \lambda), \tag{3.48}$$

$$\Xi(\mu, \lambda) = \int d\xi_1 d\xi_2 \phi_\mu(\xi_1) \phi_\lambda(\xi_2) \chi_2(\xi_1, \xi_2). \tag{3.49}$$

The function  $\chi_2(\xi_1, \xi_2)$  is nonzero provided  $\xi_1, \xi_2 \sim 1$ . Since the integration over  $\xi_1$  and  $\xi_2$  in Eq. (3.49) smears the singularities in  $\phi$  then  $\Xi$  a smooth function of  $\mu$  and  $\lambda$ . Next, a characteristic  $\omega_0$  in the integral (3.49) is  $H^{1/3}$ . Therefore  $\Xi \propto \mu^{-3/2}$  at  $\mu \gg H^{1/3}$  and  $\Xi \propto \lambda^{-3/2}$  at  $\lambda \gg H^{1/3}$ . Thus the characteristic  $\mu$  and  $\lambda$  in the integral (3.25) are less or of the order of  $H^{1/3}$ . Then we obtain

$$\begin{aligned}
F_2^{\text{pump}}(\xi, \xi) &\approx \frac{(\partial_\xi \omega_0)^2}{\omega_0} \int d\mu d\lambda \int_0^{t-t_*} d\tau \\
&\quad \exp[-i(\mu - \lambda)(t - t_* - \tau)] \Xi(\mu, \lambda) \sim H^{1/3} \frac{(\partial_\xi \omega_0)^2}{\omega_0}.
\end{aligned} \tag{3.50}$$

Since we shall obtain a slow instanton with  $H^{1/3} t_* \gg 1$ , the contribution (3.47) is larger than (3.50). That means that the pumping-produced anisotropic fluctuations give lesser contribution than deformation of an initial fluctuation. The consequence is that the tail of the vorticity PDF is insensitive to the form of the pumping correlation function and is determined solely by its zeroth moment i.e. the vorticity flux. That means universality of the statistics of strong vorticity fluctuations.

The estimation (3.47) determines the main contribution to  $F_2(\xi_1, \xi_2)$  where  $\xi_1 \sim \xi_2$ . We checked this time dependence of  $F_2$  by solving numerically Eq. (3.15) using different time-independent  $\omega_0$ . At the beginning, we have chosen  $F_2$  being determined by a typical fluctuation. Then we checked that  $F_2$  linearly grows as time runs. It is interesting to note that the antisymmetric in  $\xi_1, \xi_2$  part in  $F_2$  saturates. The behavior is in accordance with Eq. (3.15).

#### IV. INSTANTON

Substituting the expression (3.47) into Eq. (3.16) one obtains

$$\Delta\mathcal{I} \sim -H^{2/3} \int dt d\xi q \xi^{-1/3} (\partial_\xi \omega_0)^{-1} \partial_t [(t - t_*) \partial_\xi \omega_0 \ln(\omega_0 / \partial_\xi \omega_0)]. \quad (4.1)$$

Collecting the expressions (3.12, 4.1) we get finally the effective action

$$\mathcal{I}_{\text{eff}} = \int dt d\xi q \left\{ \partial_t \omega_0 - c H^{2/3} \xi^{-1/3} (\partial_\xi \omega_0)^{-1} \partial_t [(t - t_*) \partial_\xi \omega_0 \ln(\omega_0 / \partial_\xi \omega_0)] \right\} + \frac{i}{2} H \int dt d\xi_1 d\xi_2 q(\xi_1) q(\xi_2),$$

where  $c \sim 1$  and we substituted  $\chi_0(\xi_1, \xi_2) = H\theta(-\xi_1)\theta(-\xi_2)$ . By rescaling  $q$ ,  $\omega_0$  and  $H$  we can put  $c \rightarrow 1$ .

$$\begin{aligned} \mathcal{I}_{\text{eff}} = \int dt d\xi q \left\{ \partial_t \omega_0 - H^{2/3} \xi^{-1/3} (\partial_\xi \omega_0)^{-1} \partial_t [(t - t_*) \partial_\xi \omega_0 \ln(\omega_0 / \partial_\xi \omega_0)] \right\} \\ + \frac{i}{2} H \int dt d\xi_1 d\xi_2 q(\xi_1) q(\xi_2), \end{aligned} \quad (4.2)$$

We are interested in the PDF of  $\varpi$ , that is  $\omega$  coarse-grained over the scale  $R$ . In our terms, it can be written as

$$\varpi = \frac{2}{R^2} \int_0^R dr r \omega_0 \approx \omega_0 [\ln(R/L)], \quad (4.3)$$

because of the logarithmic character of the  $\omega_0$  dependence on  $r$ . Thus, we should fix  $\omega_0[\ln(R/L)] = \varpi$  at the observation time. Since we consider steady-state turbulence, the moment of measurement is arbitrary, we choose it to be  $t = 0$ . Then PDF  $\mathcal{P}(\varpi)$  can be calculated as the path integral

$$\mathcal{P} = \int \mathcal{D}\omega_0 \mathcal{D}q \exp(i\mathcal{I}_{\text{eff}}), \quad (4.4)$$

taken at the condition  $\omega_0[0, \ln(r/L)] = \varpi$  and for the fields  $\omega_0, q$  defined at negative times  $t$  [30]. The last property is explained by causality: the values of the fields at positive times cannot influence the PDF  $\mathcal{P}(\varpi)$ . Since  $\omega_0$  is fixed at a single point at  $t = 0$  the field  $q$  at  $t = 0$  satisfies  $q(\zeta) \propto \delta[\zeta - \ln(R/L)]$  that reflects the measuring procedure.

In the saddle-point approximation, we put  $\ln \mathcal{P} \approx i\mathcal{I}_{\text{eff}}^{\text{extr}}$  where  $\mathcal{I}_{\text{eff}}^{\text{extr}}$  is the extremum value of the effective action. The extremum conditions  $\delta\mathcal{I}_{\text{eff}}/\delta\omega_0 = 0 = \delta\mathcal{I}_{\text{eff}}/\delta q$  give the so-called instanton equations:

$$\partial_t \omega_0 = H^{2/3} \zeta^{-1/3} (\partial_\zeta \omega_0)^{-1} \partial_t [(t - t_*) \partial_\zeta \omega_0 \ln(\omega_0 / \partial_\zeta \omega_0)] + HQ(t), \quad (4.5)$$

$$\partial_t q + H^{2/3} \partial_\zeta \left\{ \ln(\omega_0 / \partial_\zeta \omega_0) \zeta^{-1/3} (\partial_\zeta \omega_0)^{-1} \partial_t [(t - t_*) q] \right\} = 0, \quad (4.6)$$

where  $Q(t) = -i \int d\zeta q(\zeta, t)$ . In deriving Eq. (4.6) we exploited large value of the logarithm  $\ln(\omega_0 / \partial_\zeta \omega_0) \simeq \ln|\zeta|$  so that in the main order we only account for the terms in the equations that contain the logarithm. Apart from the logarithm, the correction (4.1) depends only on the vorticity spatial derivative  $\partial_\xi \omega_0$ . As a result, the variation with respect to vorticity gives the equation (4.6), which has the form of a continuity equation, so that  $dQ/dt = 0$  in the main order. We see from Eq. (4.5) that the first term in the right-hand side is negative at  $\zeta < 0$  that is the correction (4.1) describes decrease of the vorticity due to deformation of the circular vortex by elliptic perturbations. Substituting the relation (4.5) into the expression (4.2), one finds

$$\ln \mathcal{P} \approx -\frac{H}{2} \int dt Q^2 \approx -\frac{H}{2} Q^2 |t_*|. \quad (4.7)$$

Since  $Q$  is  $t$ -independent, one readily obtains from Eq. (4.5) that  $\omega_0$  grows linearly with time,  $\omega_0(t, \xi) = \beta(\xi) \cdot (t - t_*)$ . Then we find for the factor  $\beta$

$$\beta = 2 \frac{H^{2/3}}{\xi^{1/3}} \ln \frac{\beta}{\partial_\xi \beta} + HQ. \quad (4.8)$$

Replacing here the ratio  $\beta/\partial_\xi \beta$  by  $\xi$  one obtains

$$\omega_0 = \left[ 2 \frac{H^{2/3}}{\xi^{1/3}} \ln |\xi| + HQ \right] (t - t_*), \quad (4.9)$$

$$\varpi = \left[ -2 \frac{H^{2/3}}{[\ln(L/R)]^{1/3}} \ln \ln(L/r) + HQ \right] |t_*|. \quad (4.10)$$

Then one obtains from Eq. (4.7)

$$\ln \mathcal{P} \simeq - \frac{Q^2 \varpi}{2 \{ Q - 2H^{-1/3} [\ln(L/R)]^{-1/3} \ln \ln(L/R) \}}. \quad (4.11)$$

Optimizing the expression over  $Q$  one gets

$$\ln \mathcal{P} \simeq -4H^{-1/3} [\ln(L/R)]^{-1/3} \ln \ln(L/r) \varpi \quad (4.12)$$

$$\varpi = 2 \frac{H^{2/3}}{[\ln(L/R)]^{1/3}} \ln \ln(L/r) |t_*|. \quad (4.13)$$

The value of  $\omega_0(t_*)$  does not matter with logarithmic accuracy as long it is much smaller than  $\varpi$ . The expression (4.12) leads to the final answer (1.1) where we omitted the slow factor  $\ln[\ln(L/R)]$ .

We can use the instanton solution found to check the validity of all the assumptions made in the derivation of the effective action. Remind that we consider the case  $\ln(L/R) \gg 1$ . The applicability condition of the saddle-point approximation is

$$|\varpi^3| \gg H \ln(L/R). \quad (4.14)$$

The fluctuations on the background of our instanton are indeed small: using the instanton solution we estimate  $F_2 \simeq t_* \omega_0 \xi^{-4/3} \omega_0^2 / \xi \ll \omega_0^2$  as was assumed. That justifies neglecting  $\mathcal{L}_3$  and  $n > 1$  terms in (3.9). Let us now estimate  $F_m$  for  $m > 2$  and compare it with  $F_2$ . Zero mode of (3.11) must allow cancelation of integral and non-integral terms which is possible only when the integrals in (3.11) are logarithmic. That requires  $F_m \propto r^{m-2} \propto \exp[(2-m)\xi]$ , therefore those terms are exponentially suppressed comparing to  $F_2$ . The instanton duration time is such that  $|\varpi t_*| \gg 1$  so that our instanton is “slow”, that is indeed  $\omega_0(t)$  changes slowly comparing to itself.

## V. DISCUSSION

It is illuminating to compare vorticity statistics in the direct  $2d$  cascade with the statistics of the passive scalar in a spatially smooth random flow [12, 18–20]. For a passive scalar  $\theta$  coarse-grained over a scale  $R$  less than the pumping length  $L$  one can get the asymptotic behavior of the single-point PDF in a smooth random flow by the following simple reasoning. Large values of  $\theta$  are achieved when there is no stretching for a time which is much longer than the mean stretching time  $\lambda^{-1} \ln(L/R)$ , where  $\lambda$  is the Lyapunov exponent. During that time, the passive scalar is pumped by a random forcing, i.e. it has Gaussian statistics with the linearly growing variance:

$$\mathcal{P}(\theta) \sim \int dt \mathcal{Q}(t) \exp(-\theta^2/Pt), \quad (5.1)$$

where  $\mathcal{Q}(t)$  is the probability of no stretching during time  $t$ . Stretching is correlated on the velocity timescale  $\lambda_0^{-1}$ , which is independent of  $\theta$ . For every stretching event, the scalar blob is stretched by order  $e$  and we ask for the probability that there were less than the number  $\ln Pe$  such events during  $t$ . For  $t \gg \lambda_0^{-1}$ , this is the probability of the Poisson process  $\ln \mathcal{Q}(t) \sim -\lambda_0 t + O[\ln(L/R)]$ . Doing a saddle-point integration over  $t$  we obtain the exponential tail (first suggested in [34, 35] and derived by the instanton formalism in [30, 36]):

$$\ln[\mathcal{P}(\theta)] \sim -\theta \sqrt{\lambda_0/P} + O[\ln(L/R)]. \quad (5.2)$$

In the work [21], we established some features of the direct vorticity cascade using an analogy between the vorticity  $\omega$  and passive scalar  $\theta$ . Developing that analogy further, one can propose an interpretation of the tail (1.1). For the vorticity cascade, we may use similar reasoning as for the passive scalar with the knowledge added from [21] that the stretching correlation time is the mean total stretching time  $\ln^{2/3}(L/R)/H^{1/3}$  from the scale  $R$  to  $L$ . That gives

$$\mathcal{P}(\varpi) \sim \int dt \mathcal{Q}(t, \varpi) \exp(-\varpi^2/Ht) \sim \int dt \exp \left[ -\varpi^2/Ht - tH^{1/3} \ln^{-2/3}(L/R) \right]. \quad (5.3)$$

The saddle-point integration shows that the main contribution comes from  $t \sim \varpi \ln^{1/3}(L/R)H^{-2/3}$  in agreement with (4.13), and the result  $\ln[\mathcal{P}(\varpi)] \simeq -|\omega|/|H \ln(L/R)|^{1/3}$  reproduces the dependence of (1.1). We see that vorticity is indeed like passive scalar: the stronger the fluctuation the longer it lives which leads to a sub-Gaussian PDF tail. Such tails were observed in numerical simulations [23, 40]. For quantities (like velocity) whose statistics is determined by fast events, their PDF have tails steeper than Gaussian [41].

In a finite box, coherent vortices may appear due to an inverse cascade [42–44]. The vortices have a well-defined spatial profile of the average velocity field that is explained by an interplay of the average profile time derivative (or friction) and an effective pumping related to long-correlated fluctuations [45]. An interesting question that is a subject of future investigations concerns an influence of the coherent vortices on the enstrophy cascade. One can also think about extension of our instantonic approach to the inverse (energy) cascade of the  $2d$  turbulence.

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